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BOUNDS FOR FRACTIONAL POWERS OF OPERATORS IN A HILBERT SPACE AND CONSTANTS IN MOMENT INEQUALITIES [1]

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Abstract

We derive bounds for the norms of the fractional powers of operators with compact Hermitian components, and operators having compact inverses in a separable Hilbert space. Moreover, for these operators, as well as for dissipative operators, the constants in the moment inequalities are established.

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Key Words and Phrases: linear operator, fractional powers, moment inequality, dissipative operator, compact Hermitian component, compact inverse

1. Introduction and preliminaries

Recently, many interesting papers are devoted to fractional powers of linear operators, cf. [1, 2, 4, 5, 6, 13, 15, 17] and references therein. In particular, in [19], the authors use the means of positive operators to establish Furuta-type operator monotonicity results for negative powers. In the paper [13], necessary and sufficient conditions for the validity of the generalized Hardy inequality are derived. Let A, B be bounded linear operators on a Hilbert space satisfying $0 \le B \le A$. Furuta showed the operator inequality $(A^r B^p A^r)^{\frac{1}{q}} \le A^{\frac{p+2r}{q}}$ as long as positive real numbers p, q, r satisfy

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$p + 2r \le (1+2r)q$ and $1 \le q$. In the paper [18], the author shows that this inequality is valid if negative real numbers p, q, r satisfy a certain condition. Of course we could not survey the whole subject here and refer the reader to the excellent book [14] as well as to the above listed papers and references therein.

Below we establish bounds for the norms of the fractional powers for the following operators in a separable Hilbert space: operators with compact Hermitian components and operators having compact inverses. To the best of our knowledge bounds the norms of the fractional powers of these operators were not obtained.

Let H be a separable Hilbert space with a scalar product $< .,. >$, the norm $\|.\|$ and the unit operator I. Everywhere below A is an unbounded operator in H with a domain $Dom(A)$, A^* is the adjoint of A and the set $Dom(A) \cap Dom(A^*)$ is dense in H. In addition, $A_R := (A + A^*)/2$ and $A_I := (A - A^*)/2i$; $\sigma(A)$ denotes the spectrum of A and A^{-1} is the inverse to A, $R_\lambda(A) = (A - I\lambda)^{-1}$ $(\lambda \notin \sigma(A))$ is the resolvent.

We recall that the fractional power of a sectorial operator A can be defined by the formula

$$A^{-\nu} = \frac{\sin(\pi\nu)}{\pi} \int_0^\infty t^{-\nu}(A + It)^{-1} dt \ (0 < \nu < 1), \tag{1.1}$$

cf. [12, Sect.I.5.2, formula (5.8)], [7, p.139], [11, p.71]. Furthermore, the following result (the moment inequality) for the sectorial operators is well known:

$$\|A^{-\nu}x\| \le C(\nu, w)\|A^{-w}x\|^{\nu/w}\|x\|^{1-\nu/w} \ \ (x \in H;\ 0 < \nu < w \le 1), \tag{1.2}$$

where the constant $C(\nu, w)$ does not depend on x, cf. [12, Sect. I.5.3], [7, p. 141]. In the case of positive self-adjoint operators, $C(\nu, w) = 1$, cf. [12, Sect.I.5.9]. In the general case, $C(\alpha, w)$ is unknown although it is very important for various applications. In the present paper we derive bounds for $C(\nu, w)$ for the above pointed operators as well as for dissipative operators.

We first assume that A is a dissipative operator. That is,

$$\alpha := \inf\ \sigma(A_R) > 0. \tag{1.3}$$

LEMMA 1.1. *Let condition (1.3) hold. Then* $\|A^{-\nu}\| \le 1/\alpha^\nu$ *for any* $\nu \in (0, 1)$.

P r o o f. Let $\alpha = 1$. Under this condition

$$\|(tI + A)^{-1}\| \le \frac{1}{t+1} \ \ (t \ge 0).$$

Now (1.1) implies

$$\|A^{-\nu}\| \leq \frac{\sin\ \pi\nu}{\pi} \int_0^\infty t^{-\nu}(1+t)^{-1}dt.$$

If we take in (1.1) $A = I$, we get

$$I^{-\nu} = I = I\frac{\sin\ \pi\nu}{\pi} \int_0^\infty t^{-\nu}(1+t)^{-1}dt.$$

So

$$\frac{\sin\ \pi\nu}{\pi} \int_0^\infty s^{-\nu}(1+s)^{-1}ds = 1. \tag{1.4}$$

We thus obtain the inequality $\|A^{-\nu}\| \leq 1$. Now let $\alpha \neq 1$. Then taking in the latter inequality operator A/α instead of A, we arrive at the required result. ■

Since $\|A^\mu x\| = \|A^{\mu-1}Ax\|$ $(0 < \mu < 1;\ x \in D(A))$, the previous lemma yields the following result.

COROLLARY 1.2. *Let condition (1.3) hold. Then*

$$\|A^\mu x\| \leq \frac{1}{\alpha^{1-\mu}}\|Ax\| \ \ (0 < \mu < 1;\ x \in D(A)).$$

2. The moment inequality for dissipative operators

Let us prove the following lemma.

LEMMA 2.1. *Let condition (1.3) hold. Then*

$$\|A(A + tI)^{-1}\| \leq 1 \ \ (t \geq 0). \tag{2.1}$$

P r o o f. First let $\alpha = 1$. We have $A(A+t)^{-1} = I - t(A+t)^{-1}$ and

$$(A^* + t)^{-1} + (A + t)^{-1} = (A^* + t)^{-1}(A^* + A + 2t)(A + t)^{-1}.$$

Thus

$$(I - t(A^* + t)^{-1})(I - t(A + t)^{-1}) =$$
$$I - t(A^* + t)^{-1}(A^* + A + 2t)(A + t)^{-1} + t^2(A^* + t)^{-1}(A + t)^{-1}$$
$$= I - t^2(A^* + t)^{-1}(A + t)^{-1} - t(A^* + t)^{-1}(A^* + A)(A + t)^{-1}.$$

But

$$< (A^*+t)^{-1}(A^*+A)(A+t)^{-1}x, x >=< (A^*+A)(A+t)^{-1}x, (A+t)^{-1}x > \geq 0$$

for all $x \in H$. Consequently,

$$I - t((A^* + t)^{-1}(A^* + A + 2t)(A + t)^{-1} + t^2(A^* + t)^{-1}(A + t)^{-1} \leq I.$$

This proves the lemma for $\alpha = 1$. Substituting in (2.1) A/α instead of A, we have

$$\|A(A + \alpha t)^{-1}\| \leq 1 \ \ (t \geq 0).$$

Replacing αt by t, we prove the lemma. ∎

COROLLARY 2.2. *Under condition (1.3), for any $n = 1, 2, ...$ and $w < n$, the inequality*

$$\|A^w(A + It)^{-n}\| \leq \frac{1}{(t + \alpha)^{w-n}}$$

is valid.

Indeed, by the previous lemma,

$$\|A^w(A + It)^{-n}\| = \|(A(A + It)^{-1})^w(A + It)^{w-n}\|$$
$$\leq \|(A(A + It)^{-1})^w\|\|(A + It)^{w-n}\| \leq \|(A + It)^{w-n}\|.$$

Hence, the inequality

$$\|(A + It)^{-1}\| \leq \frac{1}{t + \alpha} \tag{2.2}$$

implies the required result. ∎

Now we are in a position to formulate the main result of this section.

THEOREM 2.3. *Under condition (1.3), inequality (1.2) holds with*

$$C(\nu, w) = \frac{2 \sin(\pi\nu)}{\pi(w - \nu)^{\nu/w}\nu^{1-\nu/w}(1 - \nu)}. \tag{2.3}$$

P r o o f. First, let $\alpha = 1$. Let us use the formula

$$A^{-\nu} = \frac{n! \sin(\pi\nu)}{\pi(1 - \nu)...(n - \nu)} \int_0^\infty t^{n-\nu}(A + It)^{-n-1}dt \ \ (n = 0, 1, ...)$$

[12, Sect.1.5.3]. With $n = 1$ this gives us the equality

$$A^{-\nu} = \frac{\sin(\pi\nu)}{\pi(1 - \nu)} \int_0^\infty t^{1-\nu}(A + It)^{-2}dt.$$

Hence

$$\|A^{-\nu}x\| \leq \frac{\sin(\pi\nu)}{\pi(1 - \nu)}(J_1(\tau) + J_2(\tau)) \ \ (x \in H), \tag{2.4}$$

where

$$J_1(\tau) = \int_0^\tau t^{1-\nu}\|(A+It)^{-2}A^w\|\|A^{-w}x\|dt$$

and

$$J_2(\tau) = \int_\tau^\infty t^{1-\nu}\|(A+It)^{-2}\|\|x\|dt$$

for any $\tau > 0$. By the previous corollary, we have

$$J_1(\tau) \le \int_0^\tau t^{1-\nu}(1+t)^{-2+w}dt\|A^{-w}x\|$$

$$\le \int_0^\tau t^{w-\nu-1}dt\|A^{-w}x\| = \|A^{-w}x\|\frac{\tau^{w-\nu}}{w-\nu}$$

and

$$J_2(\tau) \le \int_\tau^\infty t^{1-\nu}(1+t)^{-2}dt\|x\| \le \|x\|\int_\tau^\infty t^{-\nu-1}dt = \frac{\|x\|}{\tau^\nu \nu}.$$

Taking τ, such that

$$\frac{\|x\|}{\tau^\nu\nu} = \|A^{-w}x\|\frac{\tau^{w-\nu}}{w-\nu}, \text{ we get } \tau = \frac{(w-\nu)^{1/w}\|x\|^{1/w}}{\|A^{-w}x\|^{1/w}\nu^{1/w}}.$$

With this τ and

$$b(\nu, w) := \frac{1}{(w-\nu)^{\nu/w}\nu^{1-\nu/w}},$$

we obtain

$$J_1(\tau) \le b(\nu, w)\|A^{-w}x\|^{\nu/w}\|x\|^{1-\nu/w}$$

and

$$J_2(\tau) \le b(\nu, w)\|A^{-w}x\|^{\nu/w}\|x\|^{1-\nu/w}.$$

Hence by (2.4) we have (2.3). So in the case $\alpha = 1$ the theorem is proved. Now replacing A by A/α we complete the proof. ∎

3. Operators with compact inverse

Let S_p $(1 \le p < \infty)$ be the Schatten-von Neumann ideal of compact operators in H with the finite norm $N_p(K) = [Trace\ (K^*K)^p]^{1/2p}$ $(K \in S_p)$. Assume that

$$A^{-1}A_I \in S_2. \tag{3.1}$$

THEOREM 3.1. *Let the conditions (3.1) and*

$$\beta(A) := \inf\ Re\ \sigma(A) > 0 \tag{3.2}$$

hold. Then

$$\|(\lambda I - A)^{-1}\| \leq \frac{e^{1/2}\tilde{\nu}_1(A)}{\rho(A,\lambda)\psi(A,\lambda)} \exp\,[\frac{\tilde{\nu}_1^2(A)}{2\psi^2(A,\lambda)}] \ \ (\lambda \notin \sigma(A)),$$

where

$$\psi(A,\lambda) := \inf_{s\in\sigma(A)} |1+\frac{\lambda}{s}| \ \ \textit{and} \ \ \tilde{\nu}_1(A) := 2\sqrt{e}\ N_2^2(A^{-1}A_I)e^{2N_2^2(A^{-1}A_I)}.$$

To prove this theorem, recall that if A is invertible and A^{-1} is compact, then there is an orthonormal basis of the triangular representation (the Schur basis) $\{e_k\}$, such that

$$A^{-1}e_j = \frac{1}{\lambda_j}e_j + \sum_{k=1}^{j-1} a_{kj}e_k,$$

where $\lambda_j, j = 1, 2, ...$ are the eigenvalues of A taken with their multiplicities, cf. [9]. So

$$A^{-1} = D + V \ \ (\sigma(A^{-1}) = \sigma(D)), \tag{3.3}$$

D is a normal operator and V is the quasinilpotent one defined by

$$De_k = \frac{1}{\lambda_k}e_k \ (k = 1, 2, ...) \text{ and } Ve_j = \sum_{k=1}^{j-1} a_{kj}e_k \ \ (j = 2, 3, ...), Ve_1 = 0.$$

In the sequel, D and V are called the diagonal and nilpotent parts of A^{-1}, respectively.

Lemma 3.2. *Let the operator $D^{-1}V$ be compact. Then it is quasinilpotent.*

P r o o f. Introduce the projection

$$P_n = \sum_{k=1}^{n} (., e_k)e_k.$$

Simple calculations show that $P_{n-1}D^{-1}VP_n = VP_n \ \ (n = 2, 3, ...)$. Now the required result is due to Lemma 7.3.3 from [8]. ■

Lemma 3.3. *For an integer $p \geq 1$, let the condition*

$$D^{-1}V \in S_{2p} \ \ (p = 1, 2, ...) \tag{3.4}$$

hold. Then

$$\|(A-\lambda I)^{-1}\| \leq \frac{e^{1/2}}{\rho(A,\lambda)} \sum_{k=0}^{p-1} \frac{\nu_p^k(A)}{\psi^k(A,\lambda)} \exp[\frac{\nu_p^{2p}(A)}{2\psi^{2p}(A,\lambda))}] \ (\lambda \notin \sigma(A)),$$

where

$$\nu_p(A) := \sum_{k=1}^{p} N_{2p}^k(D^{-1}V) e^{(1+N_{2p}^{2p}(D^{-1}V))/2}.$$

P r o o f. By (3.3) we have $A = (D+V)^{-1} = (I+D^{-1}V)^{-1}D^{-1}$. Thanks to the previous lemma, we can write out

$$(I+D^{-1}V)^{-1} = \sum_{k=0}^{\infty} (-1)^k (D^{-1}V)^k$$

and the series converges in the norm of S_p. Thus

$$A = D^{-1} + W, \tag{3.5}$$

where

$$W = A - D^{-1} = \sum_{k=1}^{\infty} (-1)^k (D^{-1}V)^k D^{-1}.$$

Since $\sigma(A^{-1}) = \sigma(D)$, by the spectrum mapping theorem, we can write out

$$\sigma(A) = \sigma(D^{-1}). \tag{3.6}$$

In addition,

$$WD = \sum_{k=1}^{\infty} (-1)^k (D^{-1}V)^k = (D^{-1}V) \sum_{k=1}^{\infty} (-1)^k (D^{-1}V)^k.$$

Moreover, thanks to Theorem 7.5.5 from [8],

$$\|(I-D^{-1}V)^{-1}\| \leq \sum_{k=0}^{p-1} N_{2p}^k(D^{-1}V) e^{(1+N_{2p}^{2p}(D^{-1}V))/2},$$

and thus

$$N_{2p}(WD) \leq N_{2p}(D^{-1}V) \|(I-D^{-1}V)^{-1}\|$$
$$\leq \sum_{k=1}^{p} N_k^{2p}(D^{-1}V) e^{(1+N_{2p}^{2p}(D^{-1}V))/2} = \nu_p(A).$$

So the operator $WD \in S_{2p}$ and according to Lemma 7.3.3 from [8] it is quasinilpotent. Furthermore,

$$(A-\lambda)^{-1} = (D^{-1} + W - \lambda) = (D^{-1} - \lambda)^{-1} (I + W(D^{-1} - \lambda)^{-1})^{-1}$$

$$= (D^{-1} - \lambda)^{-1}(I + WD(I - D\lambda)^{-1})^{-1}. \tag{3.7}$$

Since D is normal, $\|(I - D\lambda)^{-1}\| = \psi^{-1}(D, \lambda) = \psi^{-1}(A, \lambda)$ and

$$\|(D^{-1} - \lambda)^{-1}\| = \frac{1}{\rho(D^{-1}, \lambda)} = \frac{1}{\rho(A, \lambda)}.$$

Moreover, for $\lambda \notin \sigma(A)$,

$$N_{2p}(WD(I - \lambda D^{-1}V)) \leq N_{2p}(WD)\|(I - \lambda D^{-1}V)^{-1}\| \leq \frac{\nu_p(A)}{\psi(A, \lambda)}.$$

Simple calculations show that

$$P_{n-1}WD(I - \lambda D^{-1}V)^{-1}P_n = WD(I - \lambda D^{-1}V)^{-1}P_n \ \ (n = 2, 3, ...).$$

So by Lemma 7.3.3 from [8] the operator $WD(I - \lambda D^{-1}V)^{-1}$ is quasinilpotent. Again use Theorem 7.5.5 from [8]. Then,

$$\|(I + W(D^{-1} - \lambda)^{-1})^{-1}\|$$

$$\leq \sum_{k=0}^{p-1} N_{2p}^k(WD(I - \lambda D^{-1}V)^{-1}) \exp[(1 + N_{2p}^{2p}(WD(I - \lambda D^{-1}V)^{-1}))/2]$$

$$\leq e^{1/2} \sum_{k=0}^{p-1} N_{2p}^k(WD)\|(I - \lambda D^{-1}V)^{-1}\|^k \exp[\frac{1}{2} N_{2p}^{2p}(WD)\|(I - \lambda D^{-1}V)^{-1})\|^{2p}]$$

$$\leq e^{1/2} \sum_{k=0}^{p-1} \frac{\nu_p^k(A)}{\psi^k(A, \lambda)} \exp[\frac{\nu_p^{2p}(A)}{2\psi^{2p}(A, \lambda)}].$$

Now the required result is due to (3.7). ■

To estimate $N_2(D^{-1}V)$ under (3.1), note that

$$N_2^2(VD^{-1}) = \sum_{k=1}^{\infty} \|\lambda_k V e_k\|^2,$$

where e_k is the Schur basis, again. It is simple to check that

$$\|Ve_k\| \leq 2\|V_I e_k\| \ \ \ (V_I = (V - V^*)/2i).$$

By (3.3),

$$\|V_I e_k\|^2 = \|(A^{-1})_I e_k\|^2 - |Im \ \frac{1}{\lambda_k}|^2,$$

where

$$(A^{-1})_I = (A^{-1} - (A^{-1})^*)/2i = A^{-1}A_I(A^{-1})^*.$$

But $(A^{-1})^* e_k = \overline{\lambda}_k^{-1} e_k$ and therefore,

$$N_2^2(VD^{-1}) = \sum_{k=1}^{\infty} \|\lambda_k V e_k\|^2 \leq 4 \sum_{k=1}^{\infty} \|A^{-1}A_I e_k\|^2.$$

Thus we have proved the following lemma.

LEMMA 3.4. *Let condition (3.1) hold. Then* $N_2(VD^{-1}) \leq 2N_2(A^{-1}A_I)$.

The assertion of Theorem 3.1 follows from Lemmas 3.3 and 3.4, since

$$\nu_1(A) \leq \sqrt{2e}N_2(A^{-1}A_I)e^{N_2^2(A^{-1}A_I)} = \tilde{\nu}_1(A).$$

Under (3.2) we have

$$\rho(A,-t) \geq t+\beta(A) \text{ and } \psi(A,t)$$
$$= \inf\ _{s\in\sigma(A)}|1+1/st| \geq 1 \ \ (t \geq 0). \tag{3.8}$$

Now Theorem 3.1 implies

COROLLARY 3.5. *Let conditions (3.1) and (3.2) hold. Then*

$$\|(tI+A)^{-1}\| \leq \frac{\zeta_2(A)}{t+\beta(A)} \ \ (t \geq 0),$$

where

$$\zeta_2(A) = e^{1/2}\exp\,[N_2^2(A^{-1}A_I)].$$

Corollary 3.5 and formula (1.1) imply

$$\|A^{-\nu}\| \leq \zeta_2(A)\frac{\sin\ \pi\nu}{\pi}\int_0^\infty t^{-\nu}(\beta(A)+t)^{-1}dt.$$

But

$$\int_0^\infty t^{-\nu}(\beta(A)+t)^{-1}dt = \beta^{-\nu}(A)\int_0^\infty s^{-\nu}(1+s)^{-1}ds.$$

Taking into account (1.4), we thus arrive at

COROLLARY 3.6. *Let conditions (3.1) and (3.2) hold. Then*

$$\|A^{-\nu}\| \leq \frac{\zeta_2(A)}{\beta(A)^\nu} \ \ (0<\nu<1).$$

4. Operators with compact Hermitian components

In this section we assume that *Dom* $(A) =$ *Dom* (A^*) and

$$A_I = (A-A^*)/2i \in S_{2p} \ \ (p=1,2,...). \tag{4.1}$$

Standard examples of operators with Hermitian components from S_p are integro-differential operators with selfadjoint boundary conditions, as well as singular integral operators. Put

$$b_p := 2(1+\frac{2p}{\exp(2/3)ln2})\,.$$

THEOREM 4.1. *Let conditions (4.1) and (3.2) hold. Then*

$$\|A^{-\nu}\| \leq \frac{\sin(\pi\nu)}{\pi} \int_0^\infty t^{-\nu}\Lambda_p(A,t)dt \quad (0<\nu<1),$$

where

$$\Lambda_p(A,t) := \sum_{m=0}^{p-1} \frac{(b_pN_{2p}(A_I))^m}{(\beta(A)+t)^{m+1}} \ \exp\left[\frac{1}{2}+\frac{(b_pN_{2p}(A_I))^{2p}}{2(\beta(A)+t)^{2p}}\right].$$

P r o o f. We need the following result, [8, Theorem 7.9.1]: Let condition (4.1) hold. Then

$$\|R_\lambda(A)\| \leq \sum_{m=0}^{p-1} \frac{(b_pN_{2p}(A_I))^m}{\rho^{m+1}(A,\lambda)} \ \exp\left[\frac{1}{2}+\frac{(b_pN_{2p}(A_I))^{2p}}{2\rho^{2p}(A,\lambda)}\right] \quad (\lambda \notin \sigma(A)). \tag{4.2}$$

Furthermore, under (3.2), inequality (3.8) holds. Now (1.1) and(4.2) prove the theorem. ■

5. The Lyapunov norm

In the sequel we will say that an operator $-A$ is *a L^2-stable, if it generates an L^2-stable semigroup e^{-At}*. That is,

$$l(A) := 2\int_0^\infty \|e^{-At}\|^2dt < \infty. \tag{5.1}$$

Thanks to the Parseval equality

$$l(A) = \frac{1}{\pi}\int_{-\infty}^\infty \|(A+iy)^{-1}\|^2dy.$$

Put

$$W = 2\int_0^\infty e^{-A^*t}e^{-At}dt. \tag{5.2}$$

Define the scalar product

$$< x,y >_W = < Wx,y > \text{ and the norm } \|x\|_W = \sqrt{< x,x >_W}.$$

Recall that $< .,. >$ and $\|.\|$ are the scalar product and norm in H, respectively. Obviously, $\|W\| \leq l(A)$ and $\|x\|_W \leq \sqrt{l(A)}\|x\|$.

THEOREM 5.1. *Under condition (5.1), let W be defined by (5.2). Then with all $x \in H$ and $\mu,\nu \in (0,1))$ the inequalities*

$$\|A^{-\nu}x\|_W \leq \|x\|_W \leq \sqrt{l(A)}\|x\|, \tag{5.3}$$

$$\|A^\mu y\|_W \leq \|Ay\|_W \leq \sqrt{l(A)}\|Ay\| \ (y \in D(A)) \tag{5.4}$$

and

$$\|A^{-\nu}x\|_W \le C(\nu,w)\|A^{-w}x\|_W^{\nu/w}\|x\|_W^{1-\nu/w}$$
$$\le C(\nu,w)\sqrt{l(A)}\|A^{-w}x\|^{\nu/w}\|x\|^{1-\nu/w} \qquad (5.5)$$

hold where $C(\nu,w)$ is defined by (2.3).

P r o o f. First assume that A is a bounded operator in H. Recall the generalized Lyapunov theorem [3, Theorem I.5.1]: in order for the spectrum of a bounded operator A to lie in the interior of the left half-plane, it is necessary and sufficient that there exists a positive definite Hermitian operator W, such that

$$WA + A^*W = 2I. \qquad (5.6)$$

Besides W is defined by (5.2). Note that

$$W = \frac{1}{\pi}\int_{-\infty}^{\infty}(-iI\omega + A^*)^{-1}(iI\omega + A)^{-1}d\omega.$$

By (5.6) inf $\sigma(((WA)_R) = 1$, where $(WA)_R$ is the real part of WA. So (1.3) holds in the scalar product generated by W. As it is well-known, see [3, Sect.I.5]

$$\|e^{-At}x\|_W \le e^{-a_W t}\|x\|_W \ \ (t \ge 0),$$

where $a_W = \frac{1}{\|W\|}$. Now Lemma 2.1 and (5.6) imply the assertion of this theorem for bounded operators. Now let A be unbounded. Then under (5.1), there is a sequence of bounded operators A_n strongly converging to A. By the Banach-Schteinhaus theorem, $l(A_n) \to l(A)$. Letting $n \to \infty$, we arrive at the required result. ∎

Assume that (3.2) and (4.1) hold. Thanks to the Weil inequalities, $|Im\ \lambda_k(A)| \le \|A_I\|$. Then $\rho(A, iy) \ge w(y)$, where

$$w(y) \equiv \beta(A)\ (|y| \le \|A_I\|) \ \text{ and } \ w(y) = \sqrt{\beta^2(A) + y^2}\ (|y| \ge \|A_I\|).$$

Thus $l(A) \le \tilde{l}_p(A)$, where

$$\tilde{l}_p(A) := \frac{e}{\pi}\int_{-\infty}^{\infty}[\sum_{m=0}^{p-1}\frac{(b_pN_{2p}(A_I))^m}{w^{m+1}(y)}]^2\ exp\ [\frac{(b_pN_{2p}(A_I))^{2p}}{w^{2p}(y)}]dy.$$

Corollary 5.2. *Let A satisfy conditions (4.1) and (4.3) and W be defined by (5.2). Then inequalities (5.2)-(5.4) hold with $l(A) = \tilde{l}_p(A)$.*

Similarly, Theorem 3.1 gives us a bound for $C(\nu,w)$ in the case (3.1).

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